4.5 Black-Scholes-Merton Equation

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- We derive the Black-Scholes-Merton partial differential equation for the price of an option on an asset modeled as a geometric Brownian motion.

- Determine the initial capital required to perfectly hedge a short position in the option.

Portfolio Value

Consider a portfolio at time *t* valued at X(t), which invests in：

- Money market：

  Pay a constant rate of interest *r*

- Stock：

  Stock price modeled by the geometric Brownian motion

$$dS(t) = \underset{\substack{\downarrow \\ \text{平均報酬率}}}{\alpha} S(t)\,dt + \underset{\substack{\downarrow \\ \text{價格波動}}}{\sigma} S(t)\,dW(t)$$

The investor holds $\Delta(t)$ shares of stock. The position $\Delta(t)$ can be random but must be adapted to the filtration associated with the Brownian motion $W(t)$, $t \geq 0$.

$$dS(t) = \alpha S(t)\,dt + \sigma S(t)\,dW(t)$$

$$\begin{aligned} dX(t) &= \underbrace{\Delta(t)dS(t)}_{\text{Capital gain on the stock position}} + \underbrace{r(X(t) - \Delta(t)S(t))dt}_{\text{Interest earings on the cash position}} \\ &= \Delta(t)(\alpha S(t)dt + \sigma S(t)dW(t)) \\ &\quad + r(X(t) - \Delta(t)S(t))dt \\ &= rX(t)dt + \Delta(t)(\alpha - r)S(t)dt \\ &\quad + \Delta(t)\sigma S(t)dW(t) \end{aligned}$$

3 terms

- The 3 terms appearing in $dX(t)$ can be understood as follows:

(i) an average underlying rate of return r on the portfolio.

(ii) a risk premium $\alpha - r$ for investing in the stock.

(iii) a volatility term proportional to the size of the stock investment.

$$dX(t) = rX(t)dt + \Delta(t)(\alpha - r)S(t)dt + \Delta(t)\sigma S(t)dW(t)$$

Consider the discounted stock price $e^{-rt}S(t)$ and the discounted portfolio value $e^{-rt}X(t)$.

- The discounted stock price $e^{-rt}S(t)$

  According to the Itô-Doeblin formula with $f(t,x) = e^{-rt}x$:

$$\boxed{df(t,X(t)) = f_t(t,X(t))\,dt + f_x(t,X(t))\,dX(t) + \frac{1}{2}f_{xx}(t,X(t))\,dX(t)\,dX(t)}$$

$$\begin{aligned}
&d(e^{-rt}S(t)) \\
&= df(t,S(t)) \\
&= f_t(t,S(t))\,dt + f_x(t,S(t))\,dS(t) + \frac{1}{2}f_{xx}(t,S(t))\,dS(t)\,dS(t) \\
&= -re^{-rt}S(t)\,dt + e^{-rt}\,dS(t) \\
&= (\alpha - r)e^{-rt}S(t)\,dt + \sigma e^{-rt}S(t)\,dW(t)
\end{aligned}$$

$$\boxed{dS(t) = \alpha S(t)\,dt + \sigma S(t)\,dW(t)}$$

- The discounted portfolio value $e^{-rt}X(t)$:

$$
\begin{aligned}
&d\left(e^{-rt}X(t)\right) \\
&= df\left(t, X(t)\right) \\
&= f_t\left(t, X(t)\right) dt + f_x\left(t, X(t)\right) dX(t) + \frac{1}{2} f_{xx}\left(t, X(t)\right) dX(t)\, dX(t) \\
&= -re^{-rt}X(t)\, dt + e^{-rt}\, dX(t) \\
&= \Delta(t)(\alpha - r)e^{-rt}S(t)\, dt + \Delta(t)\sigma e^{-rt}S(t)\, dW(t) \\
&= \Delta(t)\, d\left(e^{-rt}S(t)\right).
\end{aligned}
$$

$$\boxed{\begin{aligned} &f(t, x) = e^{-rt}x \\ &df\left(t, X(t)\right) = f_t\left(t, X(t)\right) dt + f_x\left(t, X(t)\right) dX(t) + \frac{1}{2} f_{xx}\left(t, X(t)\right) dX(t)\, dX(t) \end{aligned}}$$

$$\boxed{dX(t) = rX(t)\, dt + \Delta(t)(\alpha - r)S(t)\, dt + \Delta(t)\sigma S(t)\, dW(t)}$$

$$\boxed{d\left(e^{-rt}S(t)\right) = (\alpha - r)e^{-rt}S(t)\, dt + \sigma e^{-rt}S(t)\, dW(t)}$$

- The change in the discounted portfolio value is solely due to change in the discounted stock price.

Option Value

- Consider a European call option that pays $(S(T)-K)^+$ at time T.
- The value of this call at any time should depend on the time and on the value of the stock price at that time.
- We let c(t,x) denote the value of the call at time t if the stock price at that time is S(t)=x.
- Our goal is to determine the function c(t, x) so we at least have a formula for the future option value.

The differential of $c(t,S(t))$

$$\boxed{df\big(t,X(t)\big) = f_t\big(t,X(t)\big)\,dt + f_x\big(t,X(t)\big)\,dX(t) + \frac{1}{2}f_{xx}\big(t,X(t)\big)\,dX(t)\,dX(t)}$$

$$\begin{aligned}
dc\big(t,S(t)\big) \\
&= c_t\big(t,S(t)\big)\,dt + c_x\big(t,S(t)\big)\,\underline{dS(t)} + \frac{1}{2}c_{xx}\big(t,S(t)\big)\,\underline{dS(t)\,dS(t)} \\
&= c_t\big(t,S(t)\big)\,dt + c_x\big(t,S(t)\big)\,\underline{\big(\alpha S(t)\,dt + \sigma S(t)\,dW(t)\big)} \\
&\qquad + \frac{1}{2}c_{xx}\big(t,S(t)\big)\underline{\sigma^2 S^2(t)\,dt}
\end{aligned}$$

$$\boxed{dS(t) = \alpha S(t)\,dt + \sigma S(t)\,dW(t)}$$

$$\begin{aligned}
&= \left[c_t\big(t,S(t)\big) + \alpha S(t)c_x\big(t,S(t)\big) + \frac{1}{2}\sigma^2 S^2(t)c_{xx}\big(t,S(t)\big)\right]\,dt \\
&\qquad + \sigma S(t)c_x\big(t,S(t)\big)\,dW(t).
\end{aligned}$$

The differential of discounted option price $e^{-rt}c(t,S(t))$

Let $f(t,x) = e^{-rt}x$

$$\boxed{df\big(t,X(t)\big) = f_t\big(t,X(t)\big)\,dt + f_x\big(t,X(t)\big)\,dX(t) + \frac{1}{2}f_{xx}\big(t,X(t)\big)\,dX(t)\,dX(t)}$$

$$\begin{aligned}
d\big(e^{-rt}c(t,S(t))\big) &= df\big(t, c(t,S(t))\big) \\
&= f_t\big(t,c(t,S(t))\big)\,dt + f_x\big(t,c(t,S(t))\big)\,dc(t,S(t)) \\
&\qquad + \frac{1}{2}f_{xx}\big(t,c(t,S(t))\big)\,dc(t,S(t))\,dc(t,S(t)) \\
&= -re^{-rt}c(t,S(t))\,dt + e^{-rt}\,dc(t,S(t)) \\
&= e^{-rt}\Big[-rc(t,S(t)) + c_t(t,S(t)) + \alpha S(t)c_x(t,S(t)) \\
&\qquad + \frac{1}{2}\sigma^2S^2(t)c_{xx}(t,S(t))\Big]\,dt + e^{-rt}\sigma S(t)c_x(t,S(t))\,dW(t).
\end{aligned}$$

$$\boxed{\begin{aligned} dc\big(t,S(t)\big) = &\Big[c_t\big(t,S(t)\big) + \alpha S(t)c_x\big(t,S(t)\big) + \frac{1}{2}\sigma^2S^2(t)c_{xx}\big(t,S(t)\big)\Big]\,dt \\ &+\sigma S(t)c_x\big(t,S(t)\big)\,dW(t)\end{aligned}}$$

- A (short option) hedging portfolio starts with some initial capital *X*(0) and invests in the stock and money market account.
- The portfolio value *X*(*t*), $\forall t \in [0,T]$ agrees with *c*(*t*,*S*(*t*)) if and only if $e^{-rt}X(t) = e^{-rt}c(t,S(t)), \quad \forall t \in [0,T]$.
- To ensure this equality we should make sure that

$d(e^{-rt}X(t)) = d(e^{-rt}c(t,S(t))), \quad \forall t \in [0,T)$ and $X(0) = c(0,S(0))$.

- Integration of above function from 0 to t

$$e^{-rt}X(t) - \cancel{X(0)} = e^{-rt}c(t,S(t)) - \cancel{c(0,S(0))}$$

$$\Rightarrow e^{-rt}X(t) = e^{-rt}c(t,S(t)), \quad \forall t \in [0,T)$$

Comparing $d(e^{-rt}X(t))$ and $d(e^{-rt}c(t,S(t)))$

$$d(e^{-rt}X(t)) = \Delta(t)(\alpha - r)\cancel{e^{-rt}}S(t)\,dt + \Delta(t)\sigma\cancel{e^{-rt}}S(t)\,dW(t)$$

$$\begin{aligned} d(e^{-rt}c(t,S(t))) = \cancel{e^{-rt}}\Big[&-rc(t,S(t)) + c_t(t,S(t)) + \alpha S(t)c_x(t,S(t)) \\ &+\frac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t))\Big]dt + \cancel{e^{-rt}}\sigma S(t)c_x(t,S(t))dW(t). \end{aligned}$$

$d(e^{-rt}X(t)) = d(e^{-rt}c(t,S(t))), \quad \forall t \in [0,T)$ holds if and only if

$$\begin{aligned} &\Delta(t)(\alpha - r)S(t)dt + \Delta(t)\sigma S(t)dW(t) \\ &= \left[-rc(t,S(t)) + c_t(t,S(t)) + \alpha S(t)c_x(t,S(t)) + \tfrac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t))\right]dt + \sigma S(t)c_x(t,S(t))dW(t) \end{aligned}$$

$$\boxed{\begin{aligned}&\Delta(t)(\alpha-r)S(t)dt+\Delta(t)\cancel{\sigma S}(t)dW(t)\\&=\left[-rc(t,S(t))+c_t(t,S(t))+\alpha S(t)c_x(t,S(t))+\tfrac{1}{2}\sigma^2S^2(t)c_{xx}(t,S(t))\right]dt+\cancel{\sigma S}(t)c_x(t,S(t))dW(t)\end{aligned}}$$

We first equate the $dW(t)$ terms, which gives :

$$\Delta(t)=c_x(t,S(t)),\quad \forall t\in[0,T)$$

This is called the delta-hedging rule.

Next equate the dt terms

$$(\cancel{\alpha}-r)S(t)c_x(t,S(t))$$

$$=-rc(t,S(t))+c_t(t,S(t))+\cancel{\alpha S(t)c_x(t,S(t))}+\tfrac{1}{2}\sigma^2S^2(t)c_{xx}(t,S(t))\quad \forall t\in[0,T)$$

$$\Longrightarrow\quad rc(t,S(t))=c_t(t,S(t))+rS(t)c_x(t,S(t))+\tfrac{1}{2}\sigma^2S^2(t)c_{xx}(t,S(t))\quad \forall t\in[0,T)$$

$$\boxed{rc(t,S(t)) = c_t(t,S(t)) + rS(t)c_x(t,S(t)) + \tfrac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t)) \quad \forall t \in [0,T)}$$

- In conclusion, we should seek a continuous function c(t, x) that is a solution to the Black-Scholes-Merton partial differential equation.

$$c_t(t,x) + rxc_x(t,x) + \frac{1}{2}\sigma^2 x^2 c_{xx}(t,x) = rc(t,x)$$

$$\forall t \in [0,T), \quad x \geq 0$$

and that satisfies the terminal condition

$$c(T,x) = (x-K)^+$$

- Suppose we have found this function 看放空部位是否能完全避險
- If an investor starts with initial capital $X(0) = c(0,S(0))$ and uses the hedge $\Delta(t) = c_x(t, S(t))$
- then $d(e^{-rt}X(t)) = d(e^{-rt}c(t,S(t)))$ will hold.

$d(e^{-rt}X(t)) = d(e^{-rt}c(t,S(t))), \quad \forall t \in [0,T)$ holds if and only if

$$\Delta(t)(\alpha - r)S(t)dt + \Delta(t)\sigma S(t)dW(t)$$
$$= \left[-rc(t,S(t)) + c_t(t,S(t)) + \alpha S(t)c_x(t,S(t)) + \tfrac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t))\right]dt + \sigma S(t)c_x(t,S(t))dW(t)$$

a solution to the Black-Scholes-Merton partial differential equation

equate the dt terms

$$rc(t,S(t)) = c_t(t,S(t)) + rS(t)c_x(t,S(t)) + \tfrac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t)) \quad \forall t \in [0,T)$$

- We see that $X(t) = c(t, S(t))$ for all $t \in [0, T)$

$d(e^{-rt}X(t)) = d(e^{-rt}c(t, S(t)))$ will hold

Integration of above function from 0 to t

$$\cancel{e^{-rt}}X(t) - \cancel{X(0)} = \cancel{e^{-rt}}c(t, S(t)) - c(0, \cancel{S(0)})$$

- Taking the limit as $t \to T$ and using the fact that both $X(t)$ and $c(t, S(t))$ are continuous, we conclude that

$$X(T) = c(T, S(T)) = (S(T) - K)^+$$

- This means that the short position has been successfully hedged.

Backward parabolic

$$\boxed{c_t(t,x) + rxc_x(t,x) + \frac{1}{2}\sigma^2 x^2 c_{xx}(t,x) = rc(t,x)}$$

- Black-Scholes-Merton PDE is a PDE of the type called Backward parabolic

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + F = 0,$$

$$B^2 - AC = 0.$$

- x represents stock price and y represents time.

Backward parabolic

- For such an equation, in addition to the terminal condition $c(T,x) = (S(T) - K)^+$, one needs boundary conditions at $x = 0$ and $x = \infty$ in order to determine the solution.

The boundary condition at $x = 0$

$$\boxed{c_t(t,x) + rxc_x(t,x) + \frac{1}{2}\sigma^2 x^2 c_{xx}(t,x) = rc(t,x) \ \ \forall t \in [0,T), \quad x \geq 0}$$

- Substituting *x* = 0 into Black-Scholes-Merton PDE, which then becomes

$$c_t(t,0) = rc(t,0)$$

and the solution is

$$c(t,0) = e^{rt}c(0,0)$$

$\frac{dy}{dx}$+P(x)y=0

y=C$e^{-\int P(x)dx}$

$\frac{dc(t,0)}{dt} - \mathrm{rc}(t,0)$=0

C(t,0)=C$e^{-\int -rdt}$=Ce^{rt}

If the initial stock price is zero, then subsequent stock prices are all zero.

S_0=0 $\quad$ S_t=0

$c(0,0)$ $\quad$ $c(t,0)$

$c(0,0)$=$\mathrm{e}^{-\mathrm{rt}}$ $c(t,0)$

$$dS(t) = \alpha S(t)\,dt + \sigma S(t)\,dW(t)$$

If S is ever zero, then dS is also zero, and therefore S can never change.

The boundary condition at $x = 0$

$$\boxed{c(t,0) = e^{rt}c(0,0)}$$

- Substituting *t* = *T* into this equation and using the fact that

  $c(T,0) = (0-K)^+ = 0$, we see that $c(0,0) = 0$ and hence

  $$c(t,0) = 0, \quad \forall t \in [0,T] \qquad \boxed{c(T,0) = e^{rT}c(0,0) = 0}$$

  this is the boundary condition at x=0.

The boundary condition at $x \to \infty$

- For large x, this call is deep in the money and very likely to end in the money. In this case, the price of the call is almost as much as the price of the forward contract.
- At expiration, the value of the forward contract is $S(T) - K$
- The value of the forward contract is $f(t,x) = x - e^{-r(T-t)}K$
- As $x \to \infty$,the function c(t,x) grows without bound. One way to specify a boundary condition at x= ∞ for the European call is

$$\lim_{x\to\infty}\left[c(t,x) - (x - e^{-r(T-t)}K)\right] = 0 \quad \forall t \in [0,T]$$

the price of the call the price of the forward contract

Black-Scholes-Merton function

- In Subsection 5.2.5, we present a derivation of this solution based on probability theory.
- In this section, however, rather than showing how to solve the equation, we shall simply present the solution and check that it works.
- The solution to the Black-Scholes-Merton equation with terminal condition and boundary conditions is

$$c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x)),\ 0 \le t < T, x > 0,$$

where

$$d_\pm(\tau,x) = \frac{1}{\sigma\sqrt{\tau}}\left[\ \ln\left(\frac{x}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)\tau\right]$$

and *N* is the cumulative standard normal distribution

$$N(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{z^2}{2}}\,dz = \frac{1}{\sqrt{2\pi}}\int_{-y}^{\infty} e^{-\frac{z^2}{2}}\,dz$$

Black-Scholes-Merton function

$$\boxed{c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x))}$$

- We shall sometimes use the notation

$$BSM(\tau, x; K, r, \sigma) = xN(d_+(\tau, x)) - Ke^{-r\tau}N(d_-(\tau, x)),$$

and call $BSM(\tau, x; K, r, \sigma)$ the Black-Scholes-Merton function.

time to expiration

current stock price

strike price

interest rate

stock volatility

Black-Scholes-Merton function

- But above function does not define $c(t,x)$ when $t = T$,nor dose it define $c(t,x)$ when $x = 0$.

$$\boxed{d_{\pm}(\tau,x) = \frac{1}{\sigma\sqrt{\tau}}\left[\ln\left(\frac{x}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)\tau\right]}$$

$\tau = T - t = 0$ ln0

$$\lim_{t \to T} c(t,x) = \text{c(T,x)} = (x - K)^+$$

$$\lim_{x \to 0} c(t,x) = \text{c(t,0)} = 0 \quad \boxed{c(t,0) = 0, \quad \forall t \in [0,T]}$$

The Greeks

- The derivatives of the function $c(t, x)$ with respect to various variables are called the Greeks.

Delta	Δ	$\partial c/\partial x$	衡量當標的物價格改變，選擇權價格的變化
Gamma	γ	$\partial^2 c/\partial x^2$	衡量當標的物價格改變，避險部位的變化
Theta	θ	$\partial c/\partial t$	
Vega	ν	$\partial c/\partial \sigma$	
Rho	ρ	$\partial c/\partial r$	

Delta

- Delta is always positive

$$\Delta = c_x(t,x) = N(d_+(T-t,x)) > 0$$

Proof. $Ke^{-r(T-t)}N'(d_-) = xN'(d_+)$.

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{x^2}{-2}}$$

$$d_- - d_+ = \frac{-\sigma^2\tau}{\sigma\sqrt{\tau}}$$

$$d_\pm(\tau,x) = \cancel{\frac{1}{\sigma\sqrt{\tau}}}\left[\ \ln\left(\frac{x}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)\tau\right]$$

$$\begin{aligned}
Ke^{-r(T-t)}N'(d_-) &= Ke^{-r(T-t)}\frac{e^{-\frac{d_-^2}{2}}}{\sqrt{2\pi}} \\
&= Ke^{-r(T-t)}\frac{e^{-\frac{(d_+-\sigma\sqrt{T-t})^2}{2}}}{\sqrt{2\pi}} \\
&= Ke^{-r(T-t)}e^{\cancel{\sigma\sqrt{T-t}}d_+}e^{-\frac{\sigma^2(T-t)}{2}}N'(d_+) \\
&= Ke^{-r(T-t)}\frac{x}{K}e^{(r+\frac{\sigma^2}{2})(T-t)}e^{-\frac{\sigma^2(T-t)}{2}}N'(d_+) \\
&= xN'(d_+).
\end{aligned}$$

Delta

- Proof. $\boxed{c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x))}$

$$
\begin{aligned}
c_x &= N(d_+) + xN'(d_+)\frac{\partial}{\partial x}d_+(T-t,x) - Ke^{-r(T-t)}N'(d_-)\frac{\partial}{\partial x}d_-(T-t,x) \\
&= N(d_+) + xN'(d_+)\frac{\partial}{\partial x}d_+(T-t,x) - xN'(d_+)\frac{\partial}{\partial x}d_+(T-t,x) \\
&= N(d_+).
\end{aligned}
$$

$$\boxed{Ke^{-r(T-t)}N'(d_-) = xN'(d_+).}$$

$$d_\pm(\tau,x) = \frac{1}{\sigma\sqrt{\tau}}\left[\ \ln(\tfrac{x}{K}) + (r \pm \tfrac{\sigma^2}{2})\tau\right]$$

$$\frac{\partial}{\partial x}d_-(T-t,x) = \frac{\partial}{\partial x}d_+(T-t,x) = \frac{1}{x\sigma\sqrt{\tau}}$$

Gamma

- Gamma is always positive

$$\gamma = c_{xx}(t,x) = \frac{N'(d_+(T-t,x))}{\sigma x\sqrt{T-t}} > 0$$

$$\boxed{c_x(t,x) = N(d_+(T-t,x)) \quad \frac{\partial}{\partial x} d_+(T-t,x) = \frac{1}{x\sigma\sqrt{\tau}}}$$

$$c_{xx}(t,x) = N'(d_+(T-t,x))\frac{\partial}{\partial x}d_+(T-t,x) = \frac{1}{\sigma x\sqrt{T-t}}N'(d_+(T-t,x)).$$

Theta

- Theta is always negative

$$\theta = c_t(t,x) = -rKe^{-r(T-t)}N(d_-(T-t,x))$$
$$-\frac{\sigma x}{2\sqrt{T-t}}N'(d_+(T-t,x)) < 0$$

Theta

- Proof. $\boxed{c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d\_(T-t,x))}$

$$
\begin{aligned}
c_t &= xN'(d_+)\frac{\partial}{\partial t}d_+(T-t,x) - rKe^{-r(T-t)}N(d_-) - \underline{Ke^{-r(T-t)}N'(d_-)}\frac{\partial}{\partial t}d_-(T-t,x)\\
&= \cancel{xN'(d_+)\frac{\partial}{\partial t}d_+(T-t,x)} - rKe^{-r(T-t)}N(d_-) - \underline{xN'(d_+)}\left[\cancel{\frac{\partial}{\partial t}d_+(T-t,x)} + \frac{\sigma}{2\sqrt{T-t}}\right]\\
&= -rKe^{-r(T-t)}N(d_-) - \frac{\sigma x}{2\sqrt{T-t}}N'(d_+).
\end{aligned}
$$

$$\boxed{Ke^{-r(T-t)}N'(d_-) = xN'(d_+).}$$

$$d_\pm(\tau,x) = \frac{1}{\sigma\sqrt{\tau}}\left[\ln\left(\frac{x}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)\tau\right]$$

$$\frac{\partial}{\partial t}d_-(T-t,x) = \frac{-\left(r - \frac{\sigma^2}{2}\right)\sigma\sqrt{T-t} - \left(\ln\left(\frac{x}{k}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)\right)\frac{-\sigma}{2\sqrt{T-t}}}{\sigma^2(T-t)}$$

$$\frac{\partial}{\partial t}d_-(T-t,x) - \frac{\partial}{\partial t}d_+(T-t,x) = \frac{\sigma^2\sigma\sqrt{T-t} + \sigma^2(T-t)\frac{-\sigma}{2\sqrt{T-t}}}{\sigma^2(T-t)}$$

- If at time *t* the stock price is *x*, then the short option hedge of calls for holding $c_x(t, x)$ shares of stock.
- The hedging portfolio value is $\boxed{c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d\_(T-t,x))}$

$$c = \underbrace{xN[d_+]}_{\text{stock}} \underbrace{- Ke^{-r(T-t)}N[d_-]}_{\text{money market}}$$

$$\boxed{\begin{aligned} c_x(t,x) &= N(d_+(T-t,x)) \\ xc_x(t,x) &= xN[d_+] \end{aligned}}$$

- The amount invested in the money market is

$$c(t,x) - xc_x(t,x) = -Ke^{-r(T-t)}N[d_-] < 0$$

- To hedge a short position in a call option, one must borrow money.

Because delta and gamma are positive, for fixed t , the function $c(t, x)$ is increasing and convex in the variable x.

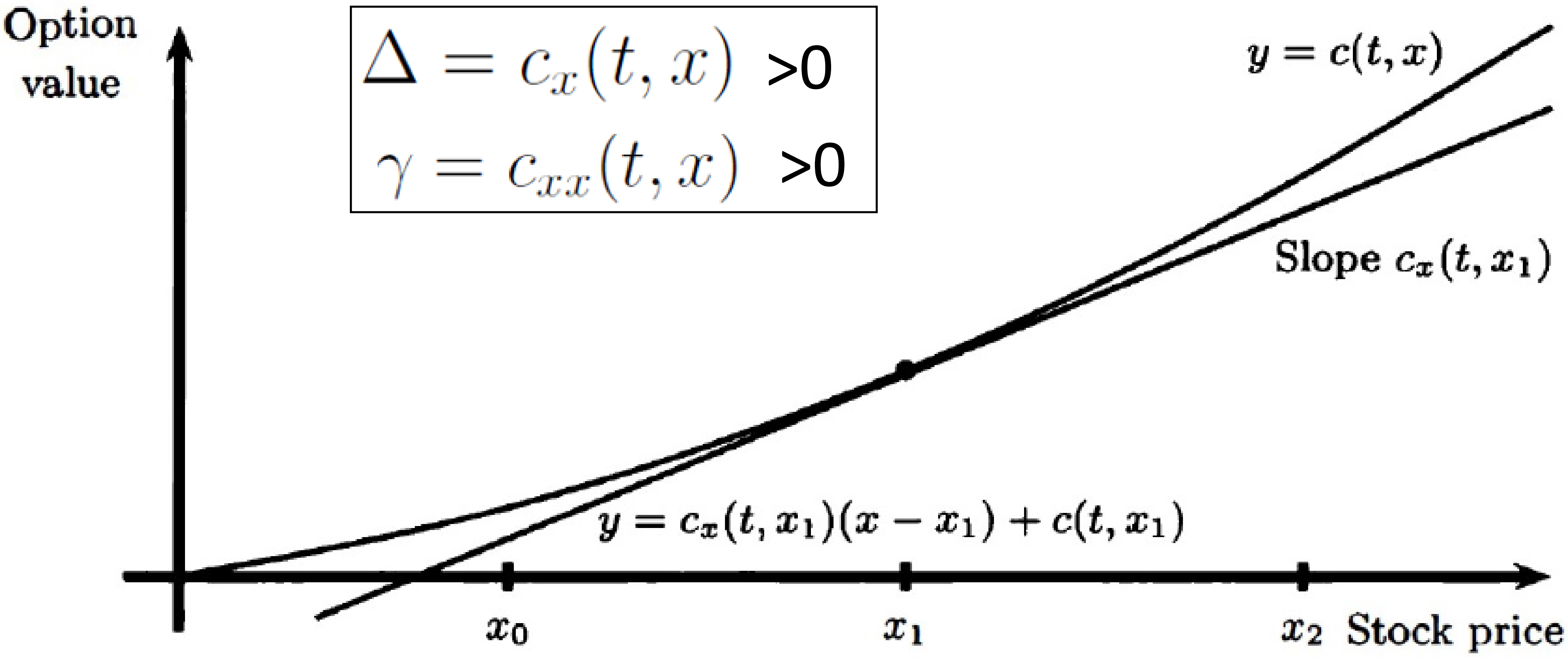


Fig. 4.5.1. Delta-neutral position.

- Suppose at time t the stock price is x_1 and we wish to take a long position in the option and hedge it.

1. Long the option for $c(t, x_1)$
2. Short $c_x(t, x_1)$ shares of stock
3. Invest in money market account

$$M = x_1 c_x(t, x_1) - c(t, x_1)$$

- The initial portfolio value

$$c(t, x_1) - x_1 c_x(t, x_1) + M = 0$$

- The portfolio we have set up is said to be delta-neutral and long gamma

- If the stock price were to instantaneously fall to x_0 and we do not change our portfolio.

$$\boxed{\begin{array}{c} c(t, x_1) - x_1 c_x(t, x_1) + M = 0 \\ M = x_1 c_x(t, x_1) - c(t, x_1) \end{array}}$$

- Total portfolio value would be

$$c(t, x_0) - x_0 c_x(t, x_1) + M = c(t, x_0) - c_x(t, x_1)(x_0 - x_1) - c(t, x_1)$$

- This is the difference at x_0 between the curve $y = c(t, x)$ and the straight line $y = c_x(t, x_1)(x - x_1) + c(t, x_1)$

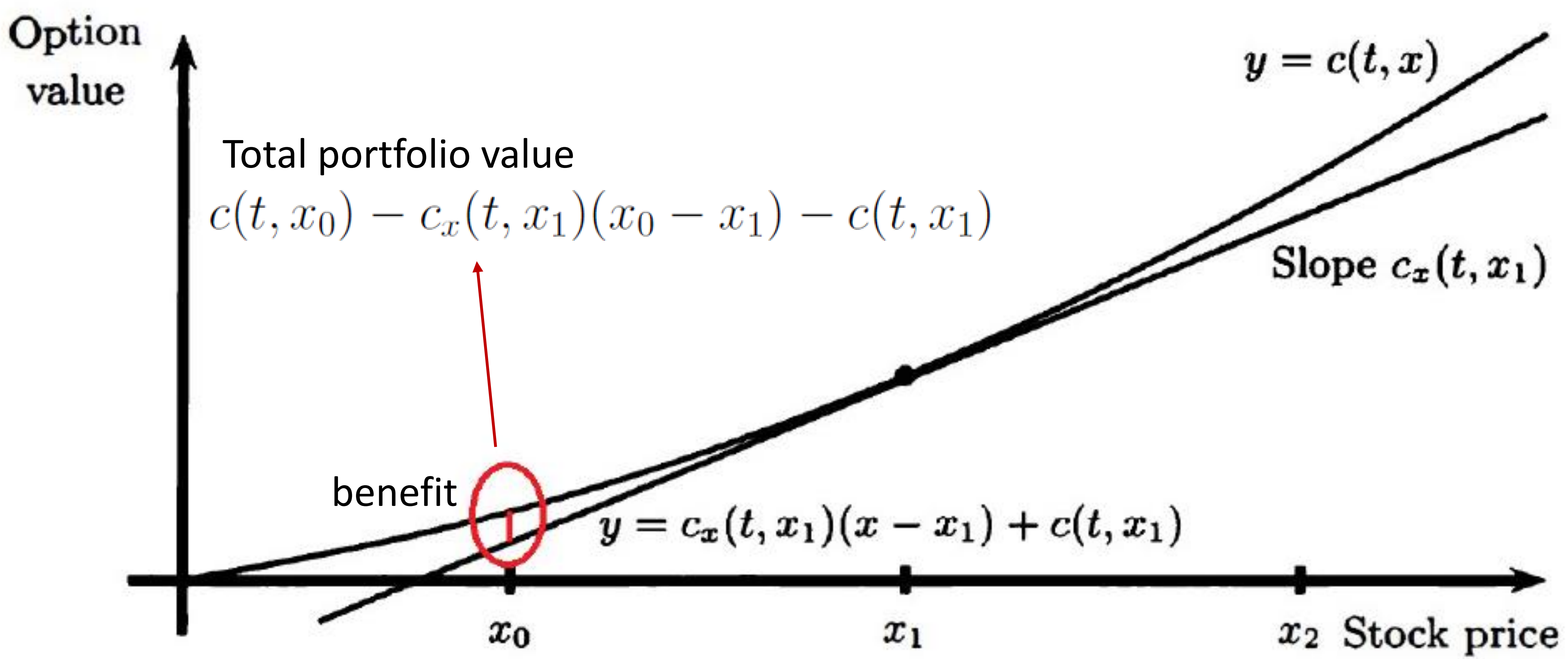


Fig. 4.5.1. Delta-neutral position.

Long Gamma

- The gamma of every option is a positive number

  → long options=long gamma

- For a delta-neutral portfolio, being long gamma means in practice that any movement in the price of the underlying product will be good news. And the bigger the price movement, the better.

- The portfolio is long gamma because it benefits from the convexity of $c(t, x)$
- If there is an instantaneous rise or an instantaneous fall in the stock price, the value of the portfolio increases.
- A long gamma portfolio is profitable in times of high stock volatility

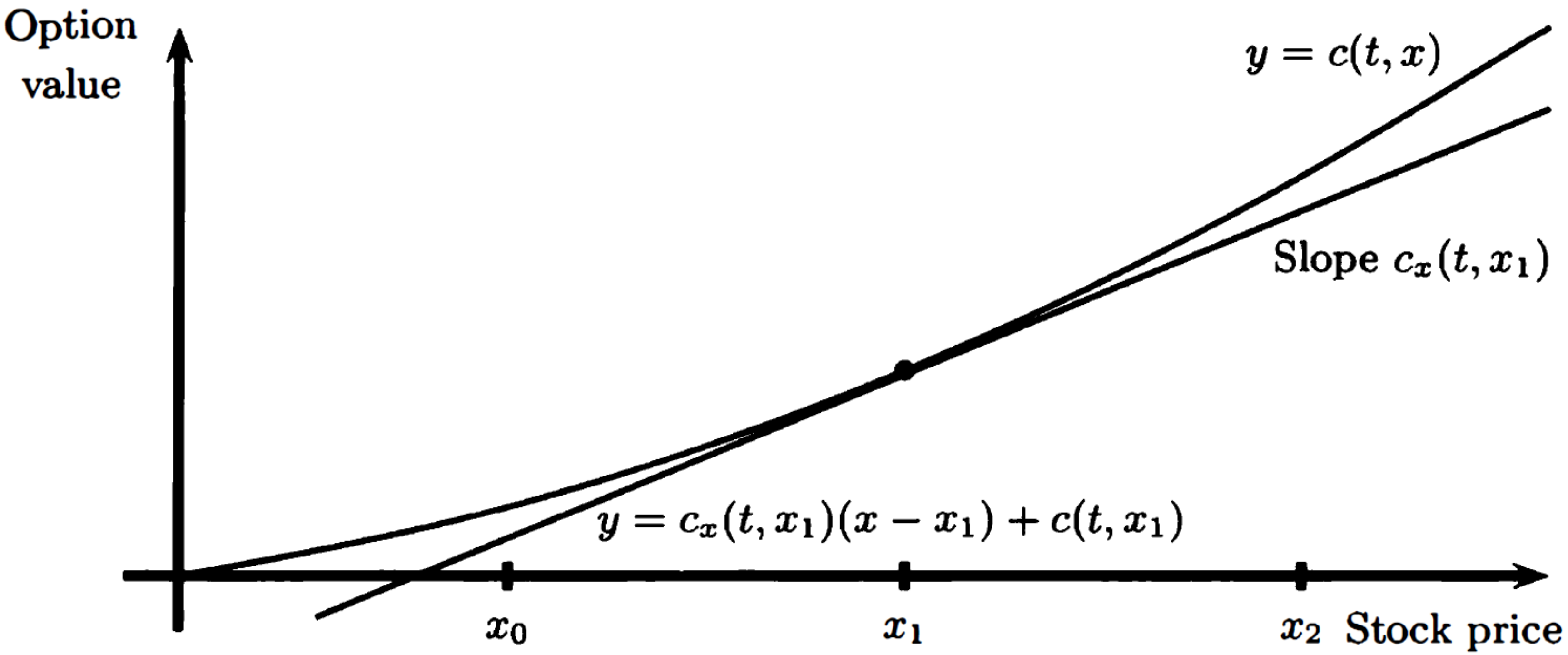


Fig. 4.5.1. Delta-neutral position.

Delta-neutral

- The delta of the stock position offsets the delta of the option position.

1. The delta of one share of stock is 1.
2. $$c_x(t,x_1)\underbrace{\times 1}_{\text{long option}} - 1 \times \underbrace{c_x(t,x_1)}_{\text{short stock}} = 0$$

- A portfolio with a delta of zero is referred as delta neutral.

Delta-neutral

- The portfolio value remains unchanged when small changes occur in the value of the underlying security.

1. The stock price goes up by \$1 ➡ a loss of $c_x(t, x_1) \times$ \$1
2. The option price will go up by $c_x(t, x_1) \times$ \$1 ➡ a gain of $c_x(t, x_1) \times$ \$1

$$\boxed{\Delta = \partial c / \partial x}$$

- Delta-neutral refers to the fact that the line is tangent to the curve
- The straight line is a good approximation to the option price for small stock price moves
- Total portfolio value is the difference between the curve and the straight line

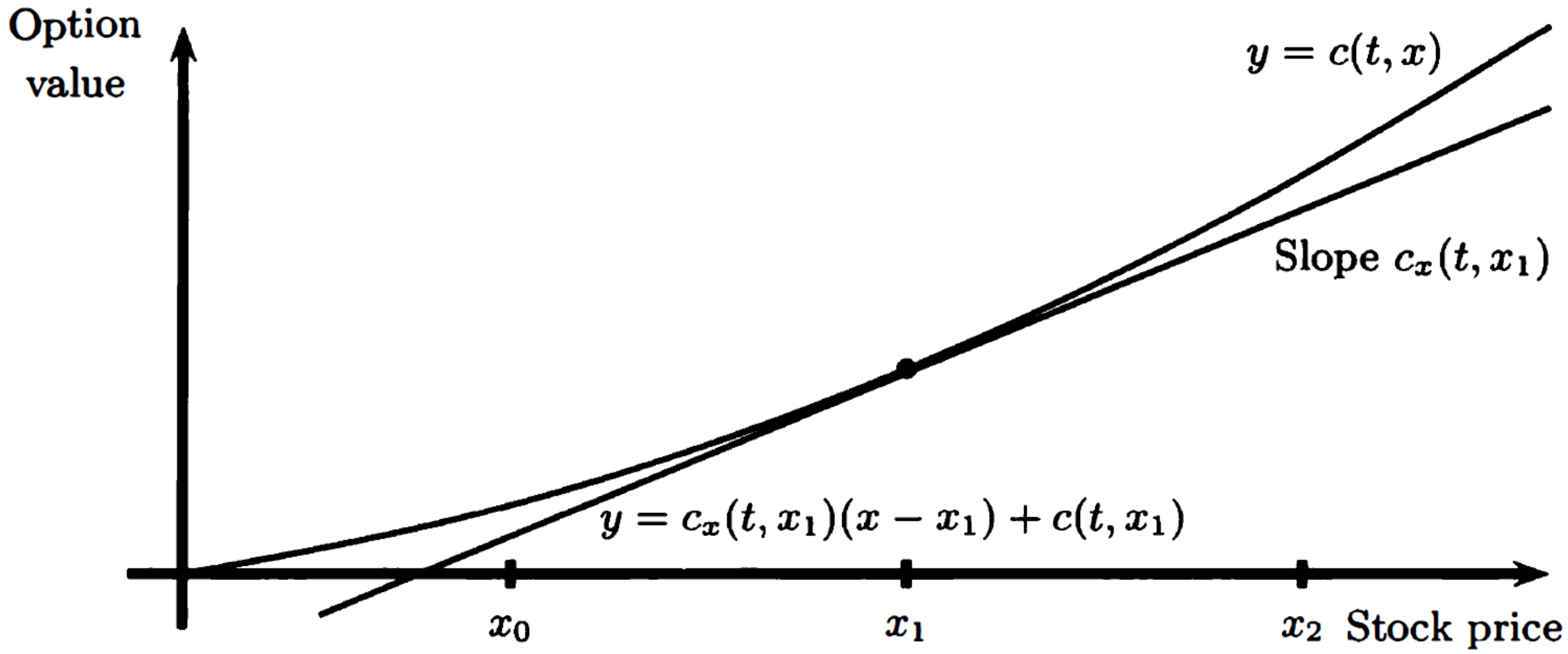


Fig. 4.5.1. Delta-neutral position.

Arbitrage opportunity

- The portfolio described above may at first appear to offer an arbitrage opportunity
- When we let time move forward

To hedge a short position in a call option, one must borrow money.

1. Both the long gamma position and the positive investment in the money market account offer an opportunity for profit
2. The curve $y = c(t, x)$ is shifts downward because theta is negative

$\boxed{\partial c / \partial t}$

- The portfolio can lose money because the curve c(t, x) shifts downward more rapidly than the money market investment and the long gamma position generate income.

Rebalancing

- The delta of an option does not remain constant.
- To keep the portfolio delta-neutral, we have to continuously rebalance our portfolio.

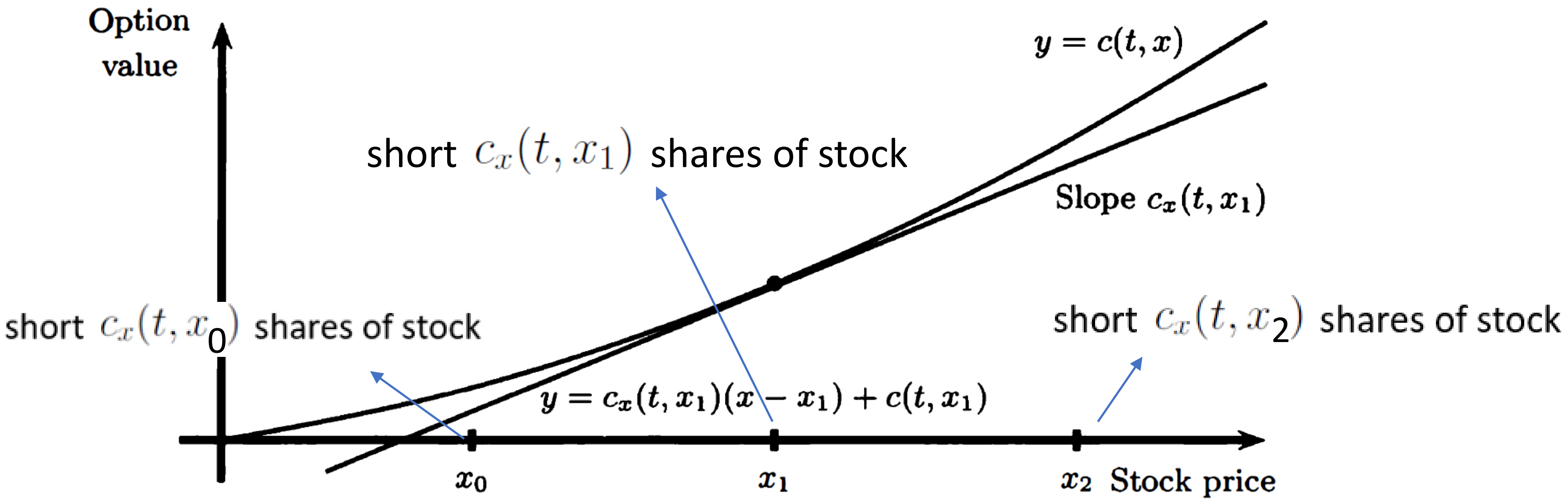


Fig. 4.5.1. Delta-neutral position.

Vega

- Actually, assets are not really geometric Brownian motions with constant volatility.
- The derivative of the option price with respect to the volatility σ is called Vega.
- Vega is positive, as volatility increases, option prices in the Black-Scholes-Merton model would increase.

Forward contract

- A forward contract with delivery price K obligates its holder to buy one share of the stock at the expiration time T in exchange for payment K
- At expiration, the value of the forward contract is $S(T) - K$
- Let $f(t, S(t))$ denote the value of the forward contract at time *t*, $t \in [0, T]$

- Risk-neutral-valuation:

  The value of the forward contract at time t is it expected value at time T in a risk-neutral world discounted at the risk-free rate of interest.

- The value of the forward contract :

$$f(t, S(t)) = e^{-r(T-t)} E_Q(S(T)-K) = e^{-r(T-t)} E_Q(S(T)) - e^{-r(T-t)} K$$

$$E_Q(S(T)) = e^{r(T-t)} s(t)$$

Expected return $\mu = r$

$$f(t, S(t)) = S(t) - e^{-r(T-t)} K, \quad \forall t \in [0, T]$$

- The agent sells this forward contract at $t = 0$ for

  $f(0, S(0)) = S(0) - e^{-rT}K$ $\boxed{f(t, S(t)) = S(t) - e^{-r(T-t)}K}$

- He can set up a static hedge, in order to protect himself.
- Static Hedge:

  A hedge that does not trade except at the initial time

1. He should purchase one share of stock by initial capital from the sale of the forward contract $S(0) - e^{-rT}K$ and the money $e^{-rT}K$ borrow from money market account.

Static hedge

$e^{-rT}K$

2. At expiration of the forward contract, he owns one share of stock and his debt to the money market account has grown to K, so his portfolio value is S(T) - K, exactly the value of the forward contract.
3. Because the agent has been able to replicate the payoff of the forward contract with a portfolio whose value at each time t is $S(t)-e^{-rT}K \times e^{rt}=S(t)-e^{-r(T-t)}K$, this must be the value at each time of the forward contract. $\boxed{f(t, S(t))= S(t) - e^{-r(T-t)}K}$

Forward price of *S*(*t*)

- The forward price of a stock at time t is defined to be the value of K that cause the forward contract at time t to have value zero.

$$S(t) - e^{-r(T-t)}K = 0$$

- The forward price at time t :

$$For(t) = e^{r(T-t)}S(t)$$

- The forward price at time t is the price one can lock in at time t for the purchase of one share of stock at time T, paying the price at time T.

- Consider a situation at $t=0$, one can lock in a price $For(0)=e^{rT}S(0)$ for buying a stock at time *T*. $\boxed{For(t)=e^{r(T-t)}S(t)}$

  Set $K=e^{rT}S(0)$ $\boxed{S(t)-e^{-r(T-t)}K=0}$
- The value of the forward contract at time t

$$f(t,S(t))=S(t)-e^{rt}S(0) \qquad \boxed{f(t,S(t))=S(t)-e^{-r(T-t)}K}$$

Put-Call Parity

- The payoff of European Options at t = T

1. European call: $c(T, S(T)) = (S(T) - K)^+$
2. European put: $p(T, S(T)) = (K - S(T))^+$
3. Forward contract: $f(T, S(T)) = S(T) - K$

- We observe that for any number x, the equation

$$x - K = (x - K)^+ - (K - x)^+$$ holds

- The equation $x - K = (x - K)^+ - (K - x)^+$ implies

$$f(T, S(T)) = c(T, S(T)) - p(T, S(T))$$

Put-Call Parity

$$f(T,S(T)) = c(T,S(T)) - p(T,S(T))$$

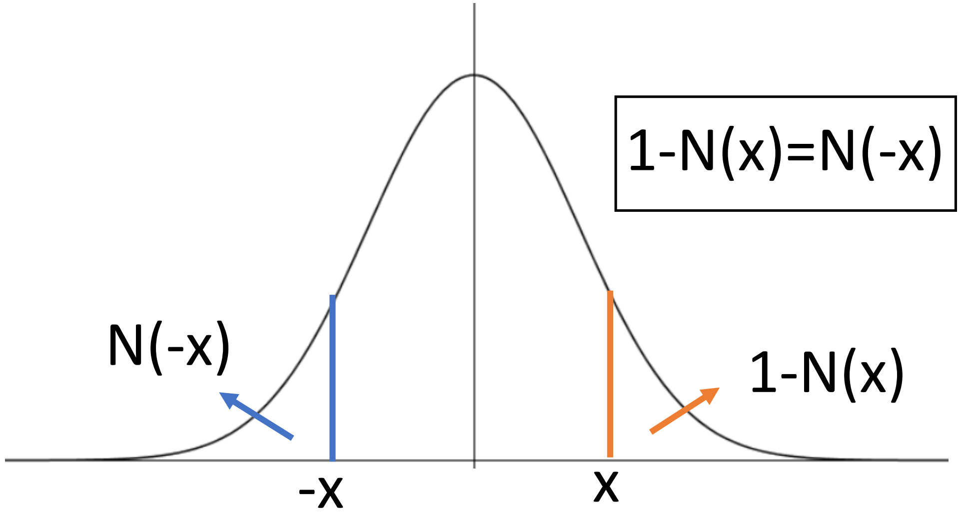
- The payoff of the forward contract agrees with the payoff of a portfolio that is long a call and short a put.
- These values must agree at all previous times

$$\underbrace{f(t,x) = c(t,x) - p(t,x)}_{\text{Put-Call Parity}}, \quad \forall x \geq 0, 0 \leq t \leq T$$

Black-Scholes-Merton put option formula

- We can use Put-Call Parity and Black-Scholes-Merton call option formula to obtain the put option formula.

$$
\begin{aligned}
p(t,x) &= c(t,x) - f(t,x) \\
&= x\big(N(d_+(T-t,x)) - 1\big) - Ke^{-r(T-t)}\big(N(d_-(T-t,x)) - 1\big) \\
&= Ke^{-r(T-t)}N(-d_-(T-t,x)) - xN(-d_+(T-t,x))
\end{aligned}
$$



$$\boxed{1-N(x)=N(-x)}$$

$$
\boxed{
\begin{aligned}
&f(t,x) = c(t,x) - p(t,x) \\
&c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x)) \\
&f(t,S(t)) = S(t) - e^{-r(T-t)}K
\end{aligned}
}
$$